Random spanning trees in random environment

(joint work with Luca Makowiec and Rongfeng Sun, NUS)

Kolmogorov meets Turing, LUISS

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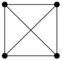
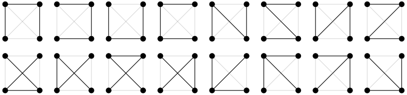
Uniform Spanning Tree

$G = (V, E)$ connected graph



A spanning tree T:
connected cycle-free subgraph of G



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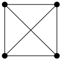

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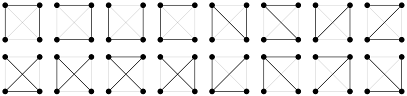

Definition

Given weights $w : E \to \mathbb{R}_+$, the *uniform spanning tree* (UST) is the random spanning tree $\mathcal{T}$ on (G, w) such that

$$P^w(\mathcal{T} = T) = \frac{1}{Z} \prod_{e \in T} w_e$$

with $Z = Z(w) := \sum_T \prod_{e \in T} w_e$.

Generating USTs

Naively Sampling

Generate all spanning trees and pick one according to its weight.

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UST on $(G, w) \leftrightarrow$ Random Walks on (G, w)

Aldous-Broder Algorithm

Run a weighted random walk until all vertices are visited. Whenever a vertex is visited for the first time, add the edge to $\mathcal{T}$.

Wilson's Algorithm

Set $\mathcal{T} = \{v\}$ for some $v \in V$. Choose $u \notin \mathcal{T}$; run weighted loop erased random walk from u until touching $\mathcal{T}$; add trajectory to $\mathcal{T}$.

Maximum Spanning Tree

Definition

Given weights $w : E \to \mathbb{R}_+$ all different, the *maximum spanning tree* (MST) is the (non-random) spanning tree T that maximizes

$$\sum_{e \in T} w_e$$

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$$\sum_{e \in T} w_e$$

How to generate it?

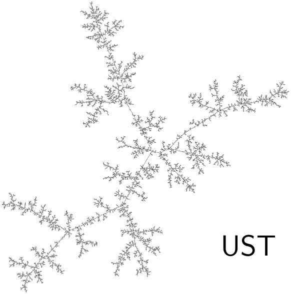
Prim's Algorithm

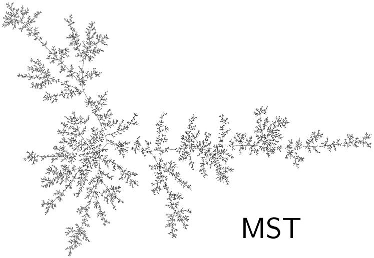
Set $\mathcal{T} = \{v\}$ for some $v \in V$. Consider all edges joining $\mathcal{T}$ to its complement and add the one with the largest weight. Iterate.

Kruskal Algorithm

Start from a forest of $|V|$ isolated vertices. Add the edge of largest weight joining two distinct components of the current forest. Iterate.

Are the UST and the MST different?





Simulations on random 3-regular graph, $|V| = 100000$, $w_e = (U[0,1])^{-1}$ (by Luca Makowiec)

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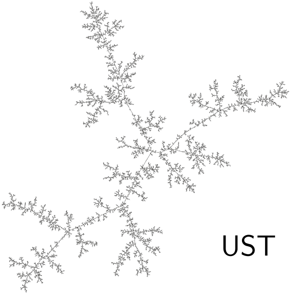
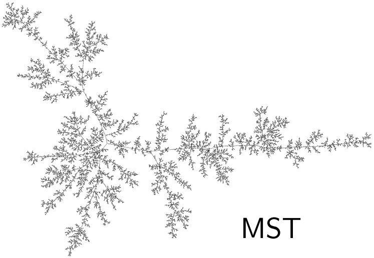


UST



MST

Simulations on random 3-regular graph, $|V| = 100000$, $w_e = (U[0,1])^{-1}$ (by Luca Makowiec)

On the complete graph $G = K_n$:

	UST	MST
Diameter	$n^{1/2}$	$n^{1/3}$
Scaling limit	$\frac{\text{UST}}{n^{1/2}} \to$ Aldous' CRT [Aldous '91]	$\frac{\text{MST}}{n^{1/3}} \to \mathcal{M}$ [Addario-B. et al. '17]

Diameter of UST

Theorem (Extension of Michaeli, Nachmias and Shalev '21)

Suppose (G, w) *is*

1. *balanced:* $$\frac{\max_{u \in V} \pi(u)}{\min_{u \in V} \pi(u)} = \frac{\max_{u \in V} \sum_v w_{(u,v)}}{\min_{u \in V} \sum_v w_{(u,v)}} \leq D;$$
2. *mixing:* $$t_{mix}(G, w) \leq n^{\frac{1}{2} - \alpha};$$
3. *escaping:* $$\sum_{t=0}^{t_{mix}} (t+1) \sup_{v \in V} p_t(v, v) \leq \theta\,.$$

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Then $\forall\varepsilon > 0$ *exists* $c = c(\varepsilon, D, \alpha, \theta)$ *such that*

$$P^w\left(c^{-1}n^{1/2} \le diam(\mathcal{T}) \le c\, n^{1/2}\right) \ge 1-\varepsilon.$$

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UST on "high-dimensional" graphs with elliptic weights have diameter $n^{1/2}$:

Examples: complete graph, expanders, high-dimensional torus...

General weights

What happens if the weights are non-elliptic?

Theorem (Makowiec, S., Sun '23)

Let $G = (V, E)$ with $|V| = n$ be either

- *an expander with max degree $\Delta < \infty$;*
- *a torus in $d \geq 5$.*

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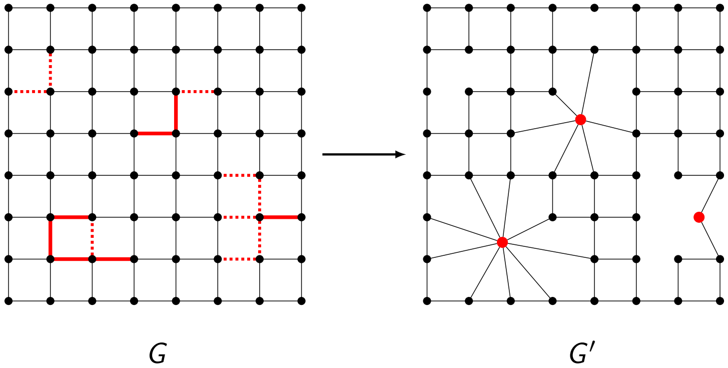
Then $\exists c > 0$ such that for all $\varepsilon > 0$

$$\mathbb{E}P^w\left(\frac{n^{1/2}}{(\varepsilon^{-1}\log n)^c} \leq diam(\mathcal{T}) \leq (\varepsilon^{-1}\log n)^c n^{1/2}\right) \geq 1 - \varepsilon\,.$$

where $\mathbb{E}$ is the expectation w.r.t. w.

Sketch of proof

1. Perform percolation on G with parameter $p = \mu(\frac{1}{A}, A)$, where A is large enough so that p is close to 1.
2. Obtain G' conditioning on the realization of $\mathcal{T}$ on closed edges:
    - delete closed edges not in $\mathcal{T}$;
    - contract closed edges in $\mathcal{T}$.



3. Verify (G', w') is balanced, mixing and escaping (use isoperimetric constant/profile) and apply [Michaeli, Nachmias, Shalev] with polylogarithmic parameters to obtain

$$\mathrm{diam}(G') \approx |V'|^{1/2} \approx n^{1/2}.$$

4. Uncontract to obtain diameter bounds on G:
   - Lower bound: Paths in G can only get longer.
   - Upper bound: Each vertex in G' consists of at most $\log n$ contracted vertices of $G \implies$ paths in G are at most $\log n$ times longer.

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Counterexample on complete graph

Take $G = K_n$ and μ heavy tailed enough. Then

$$\mathbb{E}P^w\Big(\mathcal{T} = \mathrm{MST}\Big) \xrightarrow{n\to\infty} 1\,.$$

In particular $\mathrm{diam}(\mathcal{T}) \approx n^{1/3}$.

Idea: weight of 2nd heaviest spanning tree is "super exponentially" smaller than MST.

Random Spanning Tree in Random Environment

Definition

Let $G = (V, E)$ connected graph.
Let $(\omega_e)_{e \in E}$ i.i.d. *Unif*$([0,1])$ and let $\beta \geq 0$. Assign weights

$$w_e = \mathrm{e}^{\beta \omega_e} .$$

The *Random Spanning Tree in Random Environment (RSTRE)* has law

$$P^{\omega}_{\beta}(\mathcal{T} = T) = \frac{1}{Z^{\omega}_{\beta}} \prod_{e \in T} \mathrm{e}^{\beta \omega_e} .$$

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- CASE $\beta = 0$ (Weights deterministic, Tree random)
  RSTRE is the UST. For $G = K_n$ diameter $n^{1/2}$.
- CASE $\beta = \infty$ (Weights random, Tree deterministic)
  RSTRE is the MST. For $G = K_n$ diameter $n^{1/3}$.

Can we interpolate by taking $\beta = \beta_n$?

Low Disorder

Theorem (Makowiec, S., Sun '24)

Let $G = K_n$. There exists a constant C such that if

$$\beta_n \leq C \frac{n}{\log n}$$

then for every $\delta > 0$ there exists $c = c(\delta) > 0$ such that

$$\mathbb{E}P^{\omega}_{\beta_n}\left(c^{-1} n^{1/2} \leq diam(\mathcal{T}) \leq c\, n^{1/2}\right) \geq 1 - \delta\,.$$

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Proof idea: Check the conditions of [Michaeli, Nachmias, Shalev].

- Concentration inequalities $\Longrightarrow$ balanced and isoperimetric profile;
- Cheeger inequalities + heat kernel estimates $\Longrightarrow$ mixing and escaping.

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Observations

- Extends to expanders with $\frac{\mathrm{d}_{\max}}{\mathrm{d}_{\min}} \leq C$ for $\beta_n \leq C \mathrm{d}_{\min} / \log n$.
- For $\beta_n \gg n$ proof fails: t_{mix} becomes very large (traps).

High Disorder

Theorem (Makowiec, S., Sun '24)

Let $G = K_n$. *If*

$$\beta_n \geq n^{4/3} \log n$$

then for every $\delta > 0$ *there exists* $c = c(\delta) > 0$ *such that*

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Sketch of the proof

Percolate p proportion of heaviest edges. Call $\mathcal{C}_1(p)$ the giant component.

Key Lemma

There exists $C > 0$ such that w.h.p.

$$\mathrm{diam}(\mathcal{T}) \approx \mathrm{diam}\big(\mathcal{T} \text{ on } \mathcal{C}_1(p_0)\big) \qquad \text{with} \quad p_0 = \frac{1}{n} + \frac{C \log n}{\beta_n}\,.$$

Proof Idea of Key Lemma

1. Let $u, v \in \mathcal{C}_1(p)$ and suppose (u,v) is $(p+\varepsilon)$-closed. Then

$$P^{\omega}_{\beta_n}\big((u,v)\in\mathcal{T}\big) = w_{(u,v)} R^{\omega}_{\text{eff}}(u \leftrightarrow v)$$
$$\le \mathrm{e}^{\beta_n(1-p-\varepsilon)} \cdot n\, \mathrm{e}^{-\beta_n(1-p)} = n\, \mathrm{e}^{-\beta_n \varepsilon}$$

so if $\varepsilon \ge C\frac{\log n}{\beta_n}$, this probability becomes polynomially small.

$$\big\{\mathcal{T} \text{ on } \mathcal{C}_1(p)\big\} \subseteq \mathcal{C}_1\big(p+\varepsilon\big)$$

2. Vertices outside $\mathcal{C}_1(p_0)$ with $p_0 = \frac{1}{n} + \frac{C\log n}{\beta_n}$ "hit $\mathcal{C}_1$ fast" and do not add much to the diameter.

High Disorder with Lemma

Critical window of percolation for Erdös-Rényi random graph is

$$p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}, \qquad \lambda \in \mathbb{R}.$$

If p is in the critical window, then w.h.p.

- $\mathcal{C}_1(p)$ is tree like (bounded number of cycles);
- $|\mathcal{C}_1(p)| = O(n^{2/3})$;
- $\text{diam}(\mathcal{C}_1(p)) \approx |\mathcal{C}_1(p)|^{1/2} = O(n^{1/3})$.

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So for $\beta_n \geq n^{4/3} \log n$

$$p_0 = \frac{1}{n} + \frac{C \log n}{\beta_n} \text{ is in the critical window } \Longrightarrow \mathrm{diam}(\mathcal{T}) = O(n^{1/3})\,.$$

□

Note: if $\beta_n < n^{4/3}$ then $\mathcal{C}_1(p_0)$ is not tree-like!

Future Work

Conjecture

For $G = K_n$ w.h.p.

$$diam(\mathcal{T}) \approx \begin{cases} n^{1/2}, & \beta_n \leq n \\ n^{(1-\gamma)/2}, & \beta_n = n^{1+\gamma}, \quad 0 \leq \gamma \leq \frac{1}{3} \\ n^{1/3}, & \beta_n \geq n^{4/3}. \end{cases}$$

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Idea

Show that on the slightly supercritical window

$$\mathrm{diam}(\mathcal{T}) \overset{\text{Key Lemma}}{\approx} \mathrm{diam}\left(\mathcal{T} \text{ on } \mathcal{C}_1\left(\frac{1}{n} + \frac{C \log n}{\beta_n}\right)\right)$$

$$\overset{??}{\approx} \left|\mathcal{C}_1\left(\frac{1}{n} + \frac{C \log n}{\beta_n}\right)\right|^{1/2} \overset{[DKLP14]}{\approx} \left(\frac{n^2}{\beta_n}\right)^{1/2}.$$

Thank you!